Equivariant Eilenberg-Watts theorems for locally compact quantum groups

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November 10, 2025

What is this talk about?

Let A, B be von Neumann algebras.

(Rieffel, 1974) We have the categorical equivalence

$$\mathrm{Corr}(A, B) \simeq \mathrm{Fun}(\mathrm{Rep}(B), \mathrm{Rep}(A)),$$

where

- $\mathrm{Corr}(A, B)$ denotes the category of all A-B-correspondences,
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Question: If also $A \curvearrowleft \mathbb{G}$ and $B \curvearrowleft \mathbb{G}$, where $\mathbb{G}$ is a locally compact quantum group, how can we generalize this equivalence?

Correspondences: the non-equivariant setting

Let A, B be von Neumann algebras.

Connes (1980)

An *A-B-correspondence* consists of a Hilbert space $\mathcal{H}$ together with:

1. a normal, unital $*$-representation $\pi : A \to B(\mathcal{H})$,
2. a normal, unital anti-$*$-representation $\rho : B \to B(\mathcal{H})$,

such that $\pi(a)\rho(b) = \rho(b)\pi(a)$ for all $a \in A$ and all $b \in B$. We write $\mathcal{H} = (\mathcal{H}, \pi, \rho) \in \mathrm{Corr}(A, B)$.

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- Given $\mathcal{H}, \mathcal{K} \in \mathrm{Corr}(A, B)$, we write ${}_A\mathscr{L}_B(\mathcal{H}, \mathcal{K})$ for the space of bounded A-B-bimodule maps.

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- Given $\mathcal{H}, \mathcal{K} \in \mathrm{Corr}(A, B)$, we write ${}_A\mathscr{L}_B(\mathcal{H}, \mathcal{K})$ for the space of bounded A-B-bimodule maps.
- We also write $\mathrm{Rep}(A) := \mathrm{Corr}(A, \mathbb{C})$ to denote the category of unital normal $*$-representations of A on Hilbert spaces.

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Natural objects to understand von Neumann algebras:

1. Morita equivalence.
2. Injectivity.
3. Haagerup property.
4. Property (T).
5. ...

The non-equivariant setting

- There is a natural operation (Connes fusion tensor product)

$$\mathrm{Corr}(A, B) \times \mathrm{Corr}(B, C) \to \mathrm{Corr}(A, C) : (\mathcal{H}, \mathcal{K}) \mapsto \mathcal{H} \boxtimes_B \mathcal{K}.$$

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- More concretely, if $\mathcal{H}\in\mathrm{Corr}(A,B)$ and $\mathcal{K}\in\mathrm{Corr}(B,C)$, endow the tensor product $\mathscr{L}_B(L^2(B),\mathcal{H})\odot_B\mathcal{K}$ with the semi-inner product

$$\langle x\otimes_B\xi, y\otimes_B\eta\rangle:=\langle\xi,\pi_{\mathcal{K}}(\pi_B^{-1}(x^*y))\eta\rangle.$$

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- Denote its separation-completion by $\mathcal{H}\boxtimes_B\mathcal{K}$.
- This Hilbert space becomes an A-C-correspondence through

$$\pi_\boxtimes(a)\rho_\boxtimes(c)(x\otimes_B\xi)=\pi_{\mathcal{H}}(a)x\otimes_B\rho_{\mathcal{K}}(c)\xi.$$

From correspondence to functor

- Given $\mathcal{G} \in \mathrm{Corr}(A, B)$ and $\mathcal{H} \in \mathrm{Rep}(B) = \mathrm{Corr}(B, \mathbb{C})$, form

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- In turn, this leads to the functor
$$P : \mathrm{Corr}(A, B) \to \mathrm{Fun}(\mathrm{Rep}(B), \mathrm{Rep}(A)) : \mathcal{G} \mapsto F_{\mathcal{G}}.$$

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- Then $(\mathcal{G}_F, \pi_F, \rho_F) \in \mathrm{Corr}(A, B)$.
- If $\eta = (\eta_{\mathcal{H}})_{\mathcal{H} \in \mathrm{Rep}(B)} \in \mathrm{Nat}(F, G)$, then $\eta_{L^2(B)} \in {}_A\mathscr{L}_B(\mathcal{G}_F, \mathcal{G}_G)$.

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- This leads to the functor

$$Q : \mathrm{Fun}(\mathrm{Rep}(B), \mathrm{Rep}(A)) \to \mathrm{Corr}(A, B).$$

Generators

Generators

A generator for $\mathrm{Rep}(A)$ consists of a representation $\mathcal{H} \in \mathrm{Rep}(A)$ such that for every $\mathcal{K} \in \mathrm{Rep}(A)$, there exists an index set I such that

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Fact: $\mathcal{H} \in \mathrm{Rep}(A)$ generator $\iff$ $\pi_{\mathcal{H}}$ is faithful.

Rieffel (1974)

Let $\mathcal{H} \in \mathrm{Rep}(B)$ be a generator and $F, G \in \mathrm{Fun}(\mathrm{Rep}(B), \mathrm{Rep}(A))$. Then the map

$$\mathrm{Nat}(F, G) \to \{x \in {}_A\mathscr{L}(F(\mathcal{H}), G(\mathcal{H})) \mid \forall y \in {}_B\mathscr{L}(\mathcal{H}) : xF(y) = G(y)x\}$$

given by $\eta = (\eta_{\mathcal{K}})_{\mathcal{K} \in \mathrm{Rep}(B)} \mapsto \eta_{\mathcal{H}}$ is bijective.

Rieffel (1974) - Eilenberg-Watts theorem

The functors

$$P : \mathrm{Corr}(A, B) \to \mathrm{Fun}(\mathrm{Rep}(B), \mathrm{Rep}(A)),$$
$$Q : \mathrm{Fun}(\mathrm{Rep}(B), \mathrm{Rep}(A)) \to \mathrm{Corr}(A, B)$$

are quasi-inverse to each other.

Proof.

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Proof.

It is straightforward that $Q \circ P \cong \mathrm{id}$:

$$QP(\mathcal{G}) = F_{\mathcal{G}}(L^2(B)) = \mathcal{G} \boxtimes_B L^2(B) \cong \mathcal{G}.$$

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Therefore, it suffices to show that Q is fully faithful. Thus, we need to argue that

$$\mathrm{Nat}(F, G) \to {}_A\mathscr{L}_B(\mathcal{G}_F, \mathcal{G}_G) : \eta \mapsto \eta_{L^2(B)}$$

is bijective. This is fine. $\square$

The equivariant setting

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Question: If also $A \curvearrowleft \mathbb{G}$ and $B \curvearrowleft \mathbb{G}$, where $\mathbb{G}$ is a locally compact quantum group, how can we generalize this equivalence?

Locally compact quantum groups

Kustermans-Vaes (2000)

A locally compact quantum group $\mathbb{G}$ consists of the data $(M, \Delta, \varphi, \psi)$ where

- M is a von Neumann algebra
- $\Delta : M \to M \bar{\otimes} M$ is a unital, normal, $*$-homomorphism satisfying $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.
- φ is a left invariant nsf weight on M.
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- We write $M = L^\infty(\mathbb{G})$.
- Every lcqg $\mathbb{G}$ admits a lcqg $\hat{\mathbb{G}}$ (the Pontryagin dual) such that $\hat{\hat{\mathbb{G}}} = \mathbb{G}$.

Actions on von Neumann algebras

Actions of locally compact quantum groups

Let $\mathbb{G}$ be a lcqg and A a von Neumann algebra. An action $\alpha : A \curvearrowleft \mathbb{G}$ consists of a unital, normal, isometric $*$-homomorphism $\alpha : A \to A \bar{\otimes} L^\infty(\mathbb{G})$ satisfying $(\mathrm{id} \otimes \Delta)\alpha = (\alpha \otimes \mathrm{id})\alpha$.

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Example: Let $U \in B(\mathcal{H}) \bar{\otimes} L^\infty(\mathbb{G})$ be a unitary $\mathbb{G}$-representation, i.e. U is a unitary and $(\mathrm{id} \otimes \Delta)(U) = U_{12} U_{13}$. Then

$$\alpha : B(\mathcal{H}) \to B(\mathcal{H}) \bar{\otimes} L^\infty(\mathbb{G}) : x \mapsto U(x \otimes 1)U^*$$

defines an action $B(\mathcal{H}) \curvearrowleft \mathbb{G}$.

Equivariant correspondences

Equivariant correspondences

Let $\alpha : A \curvearrowleft \mathbb{G}$ and $\beta : B \curvearrowleft \mathbb{G}$ be actions. A $\mathbb{G}$-A-B-correspondence consists of the data $(\mathcal{H}, \pi, \rho, U)$ such that $(\mathcal{H}, \pi, \rho)$ is an A-B-correspondence and $U \in B(\mathcal{H}) \bar{\otimes} L^\infty(\mathbb{G})$ is a unitary $\mathbb{G}$-representation such that

$$(\pi \otimes \mathrm{id})(\alpha(a)) = U(\pi(a) \otimes 1)U^*, \quad (\rho \otimes R)\beta(b) = U^*(\rho(b) \otimes 1)U$$

for all $a \in A$ and all $b \in B$. We write $\mathcal{H} = (\mathcal{H}, \pi, \rho, U) \in \mathrm{Corr}^{\mathbb{G}}(A, B)$.

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- We also write $\mathrm{Rep}^{\mathbb{G}}(A) := \mathrm{Corr}^{\mathbb{G}}(A, \mathbb{C})$.

Equivariant correspondences

Vaes (2001)

To every action $\alpha : A \curvearrowleft \mathbb{G}$, there is associated a canonical unitary $\mathbb{G}$-representation $U_\alpha \in B(L^2(A))\bar{\otimes}L^\infty(\mathbb{G})$ satisfying

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- Every $\mathbb{G}$-equivariant normal ucp map $\Phi : A \to B$ gives rise to a $\mathbb{G}$-A-B-correspondence.
- There is a natural operation

$$\boxtimes_B : \mathrm{Corr}^{\mathbb{G}}(A, B) \times \mathrm{Corr}^{\mathbb{G}}(B, C) \to \mathrm{Corr}^{\mathbb{G}}(A, C).$$

$$
\begin{tikzcd}
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Andruskiewitsch-Mombelli (2007)

Let H be a fd Hopf algebra and let A, B be right H-comodule algebras. Then $\mathrm{Rep}(A) \curvearrowleft \mathrm{Rep}(H)$ via

$$V \in \mathrm{Rep}(A), W \in \mathrm{Rep}(H) \rightsquigarrow V \odot W \in \mathrm{Rep}(A), \quad a(v \otimes w) = a_{(0)} v \otimes a_{(1)} w,$$

and

$$\mathrm{Corr}^{H}(A,B) \simeq \mathrm{Fun}_{\mathrm{Rep}(H)}(\mathrm{Rep}(B),\mathrm{Rep}(A)).$$

The analogue of this situation would be that if $A \curvearrowleft \mathbb{G}$, then

$$\mathrm{Rep}(A) \curvearrowleft \mathrm{Rep}(\hat{\mathbb{G}}) \cong \mathrm{Rep}(C_0^u(\mathbb{G}))$$

and

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The analogue of this situation would be that if $A \curvearrowleft \mathbb{G}$, then

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If $\mathcal{H} \in \mathrm{Rep}(A)$ and $\mathcal{K} \in \mathrm{Rep}(\hat{\mathbb{G}}) \cong \mathrm{Rep}(C_0^u(\mathbb{G}))$, we want to make sense of

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Problem: $\alpha(a) \in A \bar{\otimes} L^\infty(\mathbb{G})$, so the second leg of α does not live in $C_0^u(\mathbb{G})$ (or a completion thereof).

The module category $\mathrm{Rep}(A) \curvearrowleft \mathrm{Rep}(\hat{\mathbb{G}})$

De Commer - Krajczok (2025)

Let $\alpha : A \curvearrowleft \mathbb{G}$ be an action. There exists a unique unital, normal, isometric $*$-homomorphism

$$\alpha^u : A \to A \bar{\otimes} C_0^u(\mathbb{G})^{**}$$

such that for every $(\mathcal{H}, \pi, U) \in \mathrm{Corr}^{\mathbb{G}}(A, \mathbb{C}) = \mathrm{Rep}^{\mathbb{G}}(A)$, we have

$$(\pi \otimes \mathrm{id})\alpha^u(a) = \mathbb{U}(\pi(a) \otimes 1)\mathbb{U}^*, \quad a \in A.$$

Here, we view $\mathbb{U} \in M(\mathcal{K}(\mathcal{H}) \otimes C_0^u(\mathbb{G})) \subseteq B(\mathcal{H}) \bar{\otimes} C_0^u(\mathbb{G})^{**}$.

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α^u is a 'coaction':

$$\begin{tikzcd}
A \arrow[r, "\alpha^u"] \arrow[d, "\alpha^u"'] & A \bar{\otimes} C_0^u(\mathbb{G})^{**} \arrow[d, "\alpha^u \otimes \mathrm{id}"] \\
A \bar{\otimes} C_0^u(\mathbb{G})^{**} \arrow[r, "\mathrm{id} \otimes \tilde{\Delta}^u"'] & A \bar{\otimes} C_0^u(\mathbb{G})^{**} \bar{\otimes} C_0^u(\mathbb{G})^{**}
\end{tikzcd}$$

- Let $\mathcal{H} \in \mathrm{Rep}(A)$ and $\mathcal{K} \in \mathrm{Rep}(\hat{\mathbb{G}})$. We want to define an object $\mathcal{H} \triangleleft \mathcal{K} \in \mathrm{Rep}(A)$.

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- Consider $\phi_{\mathcal{K}} : C_0^u(\mathbb{G}) \to B(\mathcal{K})$ and its normal extension $\tilde{\phi}_{\mathcal{K}} : C_0^u(\mathbb{G})^{**} \to B(\mathcal{K})$. Then we can define

$$\pi_{\mathcal{H} \triangleleft \mathcal{K}} : A \to B(\mathcal{H} \otimes \mathcal{K}) : a \mapsto (\pi_{\mathcal{H}} \otimes \tilde{\phi}_{\mathcal{K}})(\alpha^u(a)).$$

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- We have the identity

$$\mathcal{H} \triangleleft (\mathcal{K} \boxtimes \mathcal{K}') = (\mathcal{H} \triangleleft \mathcal{K}) \triangleleft \mathcal{K}',$$

where $U_{\mathcal{K}} \boxtimes U_{\mathcal{K}'} := U_{\mathcal{K}',23} U_{\mathcal{K},13} \in B(\mathcal{H} \otimes \mathcal{K}) \bar{\otimes} L^\infty(\hat{\mathbb{G}})$.

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- We obtain the W^*-module category $\mathrm{Rep}(A) \curvearrowleft \mathrm{Rep}(\hat{\mathbb{G}})$.

From equivariant correspondence to module functor

- Let $\mathcal{G} \in \mathrm{Corr}^{\mathbb{G}}(A, B)$ be given.

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- The structure of a $\mathrm{Rep}(\hat{\mathbb{G}})$-module functor on $F_{\mathcal{G}}$ consists of a natural collection of unitaries

$$F_{\mathcal{G}}(\mathcal{H} \triangleleft \mathcal{K}) = \mathcal{G} \boxtimes_B (\mathcal{H} \triangleleft \mathcal{K}) \xrightarrow{S_{\mathcal{G},\mathcal{H},\mathcal{K}}} (\mathcal{G} \boxtimes_B \mathcal{H}) \triangleleft \mathcal{K} = F_{\mathcal{G}}(\mathcal{H}) \triangleleft \mathcal{K}$$

satisfying natural compatibilities.

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- Defining these unitaries is somehow delicate.

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$$x \otimes_B (\xi \otimes \eta) \mapsto (\mathrm{id} \otimes \phi_{\mathcal{K}})(\mathbb{U}_{\mathcal{G}})(x \otimes 1)(\mathrm{id} \otimes \phi_{\mathcal{K}})(\mathbb{U}_{\beta}^{*})(\xi \otimes \eta),$$

where $x \in \mathscr{L}_B(L^2(B), \mathcal{G}), \xi \in L^2(B)$ and $\eta \in \mathcal{K}$.

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- We then obtain the A-linear unitary

$$S_{\mathcal{G}, L^2(B), \mathcal{K}} : \mathcal{G} \boxtimes_B (L^2(B) \triangleleft \mathcal{K}) \cong \mathcal{G} \triangleleft \mathcal{K} \cong (\mathcal{G} \boxtimes_B L^2(B)) \triangleleft \mathcal{K}.$$

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- Use the fact that $L^2(B) \in \mathrm{Rep}(B)$ is a generator to find the natural unitaries

$$S_{\mathcal{G}, \mathcal{H}, \mathcal{K}} : \mathcal{G} \boxtimes_B (\mathcal{H} \triangleleft \mathcal{K}) \to (\mathcal{G} \boxtimes_B \mathcal{H}) \triangleleft \mathcal{K}, \quad \mathcal{H} \in \mathrm{Rep}(B).$$

From equivariant correspondence to module functor

- The functor $F_{\mathcal{G}} : \mathrm{Rep}(B) \to \mathrm{Rep}(A)$ together with the unitaries

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  define an element of $\mathrm{Fun}_{\mathrm{Rep}(\hat{\mathbb{G}})}(\mathrm{Rep}(B), \mathrm{Rep}(A))$.

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- In this way, we get the functor

$$\hat{P} : \mathrm{Corr}^{\mathbb{G}}(A, B) \to \mathrm{Fun}_{\mathrm{Rep}(\hat{\mathbb{G}})}(\mathrm{Rep}(B), \mathrm{Rep}(A)).$$

From module functor to equivariant correspondence

- Let $F : \mathrm{Rep}(B) \to \mathrm{Rep}(A)$ be a normal $*$-functor together with a $\mathrm{Rep}(\hat{\mathbb{G}})$-module structure:

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- Consider the unitary implementation $U_\beta \in B(L^2(B))\bar{\otimes} L^\infty(\mathbb{G})$. The condition $(\pi_B \otimes \mathrm{id})\beta(b) = U_\beta(\pi_B(b) \otimes 1)U_\beta^*$ expresses exactly that

$$U_\beta \in {}_B\mathscr{L}(L^2(B) \triangleleft (L^2(\mathbb{G}), \mathbb{I}), L^2(B) \triangleleft (L^2(\mathbb{G}), \hat{W}_{21})).$$

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- Consequently, it makes sense to define

$$U_F := S_{L^2(B),(L^2(\mathbb{G}),\hat{W}_{21})} \circ F(U_\beta) \circ S^*_{L^2(B),(L^2(\mathbb{G}),\mathbb{I})} \in B(\mathcal{G}_F \otimes L^2(\mathbb{G})).$$

From module functor to equivariant correspondence

- Then:
  1. $U_F \in B(\mathcal{G}_F)\bar{\otimes}L^\infty(\mathbb{G})$.
  2. U_F is a $\mathbb{G}$-representation.
  3. $(\mathcal{G}_F, \pi_F, \rho_F, U_F) \in \mathrm{Corr}^{\mathbb{G}}(A, B)$.

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- If $\eta = (\eta_{\mathcal{H}})_{\mathcal{H}\in\mathrm{Rep}(B)} \in \mathrm{Nat}_{\mathrm{Rep}(\hat{\mathbb{G}})}(F, G)$, meaning that we also have commutative diagrams

$$\begin{tikzcd}
F(\mathcal{H} \triangleleft \mathcal{K}) \arrow[r, "\eta_{\mathcal{H}\triangleleft\mathcal{K}}"] \arrow[d, "S_{\mathcal{H},\mathcal{K}}"'] & G(\mathcal{H} \triangleleft \mathcal{K}) \arrow[d, "S_{\mathcal{H},\mathcal{K}}"] \\
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  then $\eta_{L^2(B)} \in {}_A\mathscr{L}_B^{\mathbb{G}}(\mathcal{G}_F, \mathcal{G}_G)$.

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  then $\eta_{L^2(B)} \in {}_A\mathscr{L}_B^{\mathbb{G}}(\mathcal{G}_F, \mathcal{G}_G)$.
- In this way, we obtain the functor

$$\hat{Q} : \mathrm{Fun}_{\mathrm{Rep}(\hat{\mathbb{G}})}(\mathrm{Rep}(B), \mathrm{Rep}(A)) \to \mathrm{Corr}^{\mathbb{G}}(A, B).$$

Equivariant Eilenberg-Watts theorem

Equivariant Eilenberg-Watts theorem - Rep($\hat{\mathbb{G}}$)-module version

The functors

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are quasi-inverse to each other.

Proof (sketch).

- It is not so hard to see that $\hat{Q} \circ \hat{P} \cong \mathrm{id}$.

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Proof (sketch).

- It is not so hard to see that $\hat{Q} \circ \hat{P} \cong \mathrm{id}$.
- Hence, it suffices to show that $\hat{Q}$ is fully faithful.

Proof (continued).

- More precisely, given $F, G \in \mathrm{Fun}_{\mathrm{Rep}(\hat{\mathbb{G}})}(\mathrm{Rep}(B), \mathrm{Rep}(A))$, one needs to argue the bijectivity of

$$\mathrm{Nat}_{\mathrm{Rep}(\hat{\mathbb{G}})}(F, G) \to {}_A\mathscr{L}_B^{\mathbb{G}}(\mathcal{G}_F, \mathcal{G}_G) : \eta \mapsto \eta_{L^2(B)}.$$

Proof (continued).

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- Surjectivity is the hard part. We sketch the argument.

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- Define $\eta_2 := S^*_{L^2(B),(L^2(\mathbb{G}),\hat{W}_{21})}(\eta_1 \otimes 1)S_{L^2(B),(L^2(\mathbb{G}),\hat{W}_{21})}$, which is an A-linear morphism

$$F(L^2(B) \triangleleft (L^2(\mathbb{G}), \hat{W}_{21})) \to G(L^2(B) \triangleleft (L^2(\mathbb{G}), \hat{W}_{21})).$$

Proof (continued).

- More precisely, given $F, G \in \mathrm{Fun}_{\mathrm{Rep}(\hat{\mathbb{G}})}(\mathrm{Rep}(B), \mathrm{Rep}(A))$, one needs to argue the bijectivity of

$$\mathrm{Nat}_{\mathrm{Rep}(\hat{\mathbb{G}})}(F, G) \to {}_A\mathscr{L}_B^{\mathbb{G}}(\mathcal{G}_F, \mathcal{G}_G) : \eta \mapsto \eta_{L^2(B)}.$$

- Injectivity is clear.
- Surjectivity is the hard part. We sketch the argument.
- Let $\eta_1 \in {}_A\mathscr{L}_B^{\mathbb{G}}(\mathcal{G}_F, \mathcal{G}_G)$ be given.
- Define $\eta_2 := S^*_{L^2(B),(L^2(\mathbb{G}),\hat{W}_{21})}(\eta_1 \otimes 1)S_{L^2(B),(L^2(\mathbb{G}),\hat{W}_{21})}$, which is an A-linear morphism

$$F(L^2(B) \triangleleft (L^2(\mathbb{G}), \hat{W}_{21})) \to G(L^2(B) \triangleleft (L^2(\mathbb{G}), \hat{W}_{21})).$$

- Similarly, define an A-linear morphism

$$\eta_3 : F(L^2(B) \triangleleft (L^2(\mathbb{G}) \boxtimes L^2(\mathbb{G}))) \to G(L^2(\mathbb{G}) \triangleleft (L^2(\mathbb{G}) \boxtimes L^2(\mathbb{G})))$$

by $\eta_3 := S^*_{L^2(B)\triangleleft L^2(\mathbb{G}),L^2(\mathbb{G})}(\eta_2 \otimes 1)S_{L^2(B)\triangleleft L^2(\mathbb{G}),L^2(\mathbb{G})}$.

Proof (continued)

- For every $x \in {}_B\mathscr{L}(L^2(B) \triangleleft (L^2(\mathbb{G}) \boxtimes L^2(\mathbb{G})))$, we have $G(x)\eta_3 = \eta_3 F(x)$.

Proof (continued)

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- $L^2(B) \triangleleft (L^2(\mathbb{G}) \boxtimes L^2(\mathbb{G}))$ is a generator for $\mathrm{Rep}(B)$, so there is a unique natural transformation $\eta : F \implies G$ satisfying $\eta_{L^2(B)\triangleleft(L^2(\mathbb{G})\boxtimes L^2(\mathbb{G}))} = \eta_3$.

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- There is an isometric intertwiner $L^2(\mathbb{G}) \hookrightarrow L^2(\mathbb{G}) \boxtimes L^2(\mathbb{G})$. From this, we can conclude that $\eta_2 = \eta_{L^2(B)\triangleleft L^2(\mathbb{G})}$.

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- We now want to argue that the diagram

$$\begin{tikzcd}
F(\mathcal{H} \triangleleft \mathcal{K}) \arrow[r, "\eta_{\mathcal{H}\triangleleft\mathcal{K}}"] \arrow[d, "s_{\mathcal{H},\mathcal{K}}"'] & G(\mathcal{H} \triangleleft \mathcal{K}) \arrow[d, "s_{\mathcal{H},\mathcal{K}}"] \\
F(\mathcal{H}) \triangleleft \mathcal{K} \arrow[r, "\eta_{\mathcal{H}}\triangleleft 1"] & G(\mathcal{H}) \triangleleft \mathcal{K}
\end{tikzcd}$$

  commutes.

Proof (continued).

- By the definition of η_3, the diagram commutes when $\mathcal{H} = L^2(B) \triangleleft L^2(\mathbb{G}) \in \mathrm{Rep}(B)$ and $\mathcal{K} = L^2(\mathbb{G}) \in \mathrm{Rep}(\hat{\mathbb{G}})$.

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- Since $L^2(B) \lhd L^2(\mathbb{G})$ generates $\operatorname{Rep}(B)$ and since $L^2(\mathbb{G})$ generates the full subcategory $\operatorname{Rep}(L^\infty(\mathbb{G})) \subseteq \operatorname{Rep}(\hat{\mathbb{G}})$, we conclude that the diagram commutes when $\mathcal{H} \in \operatorname{Rep}(B)$ and $\mathcal{K} \in \operatorname{Rep}(L^\infty(\mathbb{G}))$.

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- If $\mathcal{K} \in \mathrm{Rep}(\mathbb{G})$, then $\mathcal{K} \boxtimes L^2(\mathbb{G}) \in \mathrm{Rep}(L^\infty(\mathbb{G}))$. By a diagram chase, we then find that the diagram commutes when $\mathcal{H} \in \mathrm{Rep}(B)$ and $\mathcal{K} \in \mathrm{Rep}(\hat{\mathbb{G}})$.

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- By the definition of η_3, the diagram commutes when $\mathcal{H} = L^2(B) \triangleleft L^2(\mathbb{G}) \in \mathrm{Rep}(B)$ and $\mathcal{K} = L^2(\mathbb{G}) \in \mathrm{Rep}(\hat{\mathbb{G}})$.
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- Conclusion: $\eta \in \mathrm{Nat}_{\mathrm{Rep}(\hat{\mathbb{G}})}(F, G)$.
- Finally, by the definition of η_2, we can then conclude that $\eta_{L^2(B)} = \eta_1$.

$\square$

Corollaries

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- $\mathrm{Rep}(A)$ and $\mathrm{Rep}(B)$ are equivalent as $\mathrm{Rep}(\hat{\mathbb{G}})$-module W^*-categories if and only if (A,α) and (B,β) are $\mathbb{G}$-equivariantly Morita equivalent, i.e. there exists a $\mathbb{G}$-A-B-correspondence $(\mathcal{G},\pi,\rho,U)$ such that π and ρ are faithful and $\pi(A)' = \rho(B)$.

The module category $\mathrm{Rep}^{\mathbb{G}}(A) \curvearrowleft \mathrm{Rep}(\mathbb{G})$

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- Let $A \curvearrowleft \mathbb{G}$ be an action. We define a functor

$$\triangleleft : \mathrm{Rep}^{\mathbb{G}}(A) \times \mathrm{Rep}(\mathbb{G}) \to \mathrm{Rep}^{\mathbb{G}}(A)$$

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- Then $\mathcal{H} \triangleleft (\mathcal{K} \boxtimes \mathcal{K}') = (\mathcal{H} \triangleleft \mathcal{K}) \triangleleft \mathcal{K}'$ for $\mathcal{H} \in \mathrm{Rep}^{\mathbb{G}}(A)$ and $\mathcal{K}, \mathcal{K}' \in \mathrm{Rep}(\mathbb{G})$.

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- We obtain the W^*-module category $\mathrm{Rep}^{\mathbb{G}}(A) \curvearrowleft \mathrm{Rep}(\mathbb{G})$.

From equivariant correspondence to module functor

- Given $\mathcal{G} \in \mathrm{Corr}^{\mathbb{G}}(A, B)$, we have the induced normal $*$-functor

$$F_{\mathcal{G}} : \mathrm{Rep}^{\mathbb{G}}(B) \to \mathrm{Rep}^{\mathbb{G}}(A) : \mathcal{H} \mapsto \mathcal{G} \boxtimes_B \mathcal{H}.$$

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- It becomes a $\mathrm{Rep}(\mathbb{G})$-module functor for the unitaries

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- In this way, we obtain the functor

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- Then $(\mathcal{G}_F, \pi_F, \rho_F, U_F) \in \mathrm{Corr}^{\mathbb{G}}(A, B)$.
- We obtain the functor

$$Q : \mathrm{Fun}_{\mathrm{Rep}(\mathbb{G})}(\mathrm{Rep}^{\mathbb{G}}(B), \mathrm{Rep}^{\mathbb{G}}(A)) \to \mathrm{Corr}^{\mathbb{G}}(A, B).$$

Equivariant Eilenberg-Watts theorem

Equivariant Eilenberg-Watts theorem - Rep($\mathbb{G}$)-module version

The functors

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$$Q : \mathrm{Fun}_{\mathrm{Rep}(\mathbb{G})}(\mathrm{Rep}^{\mathbb{G}}(B), \mathrm{Rep}^{\mathbb{G}}(A)) \to \mathrm{Corr}^{\mathbb{G}}(A, B)$$

are quasi-inverse to each other.

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$$Q : \mathrm{Fun}_{\mathrm{Rep}(\mathbb{G})}(\mathrm{Rep}^{\mathbb{G}}(B), \mathrm{Rep}^{\mathbb{G}}(A)) \to \mathrm{Corr}^{\mathbb{G}}(A, B)$$

are quasi-inverse to each other.

- Proof strategy is very similar as before.

Equivariant Eilenberg-Watts theorem

Equivariant Eilenberg-Watts theorem - Rep($\mathbb{G}$)-module version

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are quasi-inverse to each other.

- Proof strategy is very similar as before.
- In particular, also $\mathrm{Rep}^{\mathbb{G}}(A)$ is equivalent with $\mathrm{Rep}^{\mathbb{G}}(B)$ as Rep($\mathbb{G}$)-module W^*-categories if and only if (A, α) and (B, β) are $\mathbb{G}$-equivariantly Morita equivalent.

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- The action $\mathrm{Rep}^{\mathbb{G}}(A) \curvearrowleft \mathrm{Rep}(\mathbb{G})$ restricts to an action $\mathrm{Rep}(A \rtimes_\alpha \mathbb{G}) \curvearrowleft \mathrm{Rep}(\mathbb{G})$.
- We also have

$$\mathrm{Corr}^{\mathbb{G}}(A, B) \simeq \mathrm{Fun}_{\mathrm{Rep}(\mathbb{G})}(\mathrm{Rep}(B \rtimes_\beta \mathbb{G}), \mathrm{Rep}(A \rtimes_\alpha \mathbb{G})).$$

Thanks for your attention!